International Journal of Theoretical Physics, Vol. 39, No. 2, 2000

Weyl Spinor and Solution of Massless Free Field Equations

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Received May 28, 1999

The massless field equations for arbitrary spin in curved space-time are reconsidered. The general solution of the field equation in Robertson–Walker space-time that was previously determined is briefly discussed after explicitly showing that the Weyl spinor vanishes. The case of the Lemaître–Tolman–Bondi space-time is studied in detail. The general expression of the corresponding Weyl spinor is obtained and some particular situations exploited. The spin-3/2 and spin-2 massless field equations are solved explicitly. The solutions are simplified by the existence of nontrivial algebraic constraints. The angular part of the equations is separated by the usual separation method and integrated directly. The other equations that are not separated in the radial and time dependence are reduced to a simple form. The results obtained are extended, as a consequence of previous results, to the case of arbitrary spin. The solution of the general case essentially reduces to the treatment of spin 3/2 and spin 2.

1. INTRODUCTION

The massless free field equations in curved space-time are of interest not only on physical grounds, but also because the space-time curvature in general implies restrictions on the field components. The object of this paper is to find explicit solutions of the massless free field equations in concrete examples of space-time. As usual, the Newman and Penrose (1962) spinorial formalim is the best tool to formulate these equations. Accordingly, the massless free field equations for field of spin $s = (n + 1)/2$ are written

$$\nabla_{AA'}\, \phi^{A}_{A_1A_2\,\dots\,A_n} = 0, \qquad \phi_{AA_1A_2\dots A_n} = \phi_{(AA_1A_2\dots A_n)} \tag{1}$$

As is well known, these equations are consistent for $s = 1/2$ and $s = 1$ in a general curved space-time, while for higher spin values they must satisfy

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0020-7748/00/0200-0377$18.00/0

consistency algebraic conditions. Indeed, by applying the covariant spinorial operator $\nabla^{A_1A'}$ to Eqs. (1) one concludes (Buchdahl, 1958, 1962; Plebański, 1965; Penrose and Rindler, 1984) that the field must satisfy, for $n \geq 1$ ($s \geq 3/2$), the n conditions

$$(n - 1)\phi_{AA_1M(A_2A_3\ldots A_{n-1}}\Psi^{AA_1M}_{A_n)} = 0 \qquad (2)$$

Ψ_{ABCD} is the conformal Weyl spinor. Therefore in a general space-time the $n + 2$ independent components of the field are, in principle, drastically reduced to two by the constraints (2). On the other hand, Eqs. (2) are automatically satisfied for conformally Minkoskian space-times or for $\phi_{AA_1A_2\ldots A_n}$ the Weyl spinor Ψ_{ABCD}. A discussion of intermediate situations according to the Petrov-type classification of the Weyl spinor can be found in Bell and Szekeres (1972) and references therein. The existence of the constraints (2) has, however, some advantages. In many cases they imply that some of the spinor field components have to be chosen to be zero. After all, this is a simplification of Eq. (1), whose solution could be otherwise very difficult to obtain in general.

In the following we study Eq. (1) in the Robertson–Walker (RW) and Lemaître–Tolman–Bondi (LTB) space-times. The solution of that equation in the RW space-time, which is well known to be a conformally flat space-time (see, e.g., Penrose and Rindler, 1984), has already been obtained. The result was performed by generalizing the explicit solution of the case $s = 2$ (Zecca, 1996). As will be done in the next section, it is, however, of some interest to calculate one of the curvature spinors, in the RW space-time, that leads in a straighforward way to the vanishing of the Weyl spinor.

Equations (1) and (2) are then studied in the LTB space-time that is known to be of Petrov type D (Krasiński, 1997, and references therein) [for the LTB cosmological model see also Zecca (1993)]. By the use of a suitable null tetrad frame the general form of the Weyl spinor is determined. Some examples of LTB space-time with vanishing Weyl spinor are briefly discussed. Then Eqs. (1) and (2) are studied explicitly for $s = 3/2$, $s = 2$. In both cases it is possible to separate the angular part of the solution, which can easily be integrated. Knowledge of the r, t dependence of the field solution is reduced to the study of differential equations not separated, in general, in the r, t dependence. Even if these equations can be reduced to a simple form, their solution requires the explicit form of the metric coefficient.

By following considerations of a previous paper (Zecca, 1996), the results relative to the solution of the massless free field equations can be generalized, by induction, to arbitrary spin values. This because the structure of the Weyl spinor together with the constraints (2) implies that only two or three independent components of the field are nonzero according to whether the spin is half-integer or integer. In turn this implies that the equations of the general case can be integrated as for $s = 3/2$ or $s = 2$.

2. ROBERTSON–WALKER SPACE-TIME

Even if the geometry of the RW space-time model is well known (Penrose and Rindler, 1984; Krasiński, 1997), it is of some interest to show the vanishing of the Weyl spinor by a direct computation. To that end, associated to the metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1 - ar^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right], \qquad a = 0, \pm 1 \tag{3}$$

we consider the null tetrad frame used in Zecca (1996), whose corresponding σ-matrices have the form

$$\sigma^t_{AA'} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^r_{AA'} = \frac{(1 - ar^2)^{1/2}}{2R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^\theta_{AA'} = \frac{1}{2R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^\varphi_{AA'} = \frac{i}{2rR\sin\theta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

The Weyl spinor Ψ_{ABCD} defined by the gravitational spinor X_{ABCD}

$$\Psi_{ABCD} := X_{(ABCD)} \equiv X_{A(BCD)} \tag{5}$$

can then be expressed by the σ-matrices since

$$X_{ABCD} := \frac{1}{4}\, \sigma^a_{AX'}\sigma^{bX'}_{B}\sigma^c_{CY'}\sigma^{dY'}_{D} R_{abcd} \tag{6}$$

R_{abcd} is the Riemann curvature tensor and the identifications $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$ hold. By applying tabulated results for spherically symmetric space-times, one immediatly gets from the metric (3) (Cahill and McVittie, 1970) the independent nonzero component of the Riemann tensor ($\dot{R} = \partial R/\partial t$)

$$R_{2323} = -r^4R^2(a + \dot{R}^2)\sin^2\theta$$

$$R_{1212} = \frac{R_{3131}}{\sin^2\theta} = -\frac{r^2R^2}{1 - ar^2}(a + \dot{R}^2)$$

$$R_{1010} = \frac{R\ddot{R}}{1 - ar^2} \tag{7}$$

$$R_{2020} = \frac{R_{3030}}{\sin^2\theta} = r^2 R\ddot{R}, \qquad a = 0, \pm 1$$

By using the expressions (4) and (7), one can then calculate the curvature spinor X_{ABCD} in (6). To that end it is useful to tabulate expressions of the form $\sigma^a_{AX'}\sigma^{bX'}_B$. After this operation and some algebraic computations one finally gets

$$X_{ABCD} = \frac{a + \dot{R}^2 + R\ddot{R}}{4}\left[r^2\sigma^\theta_{AB}\sigma^\theta_{CD} - \frac{1}{R^2}\sigma^t_{AB}\sigma^t_{CD} + \frac{1}{1 - ar^2}\sigma^r_{AB}\sigma^r_{CD}\right] \tag{8}$$

By defining as usual

$$\phi_h \equiv \phi_{AA_1A_2\ldots A_n} \Leftrightarrow A + A_1 + A_2 \ldots + A_n = h, \qquad h = 0, 1, 2, \ldots, n + 1 \tag{9}$$

from (8) and from $\Psi_{ABCD} \equiv X_{A(BCD)}$, it is then easy to show that

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0 \tag{10}$$

The RW space-time is not only locally conformal to Minkowski space, as implied by Eq. (10), but it can be shown that it is also conformally flat (Penrose and Rindler, 1984). The constraints (2) have no effect and Eq. (1) have to be integrated in their generality. This has been already done in Zecca (1996), where the equations have been separated for arbitrary spin value by applying the usual separation method. All the separated equations have been integrated in general, except the radial equations relative to the open and closed space-time case.

3. LTB SPACE-TIME: THE WEYL SPINOR

We consider now the general LTB space-time of metric

$$ds^2 = dt^2 - e^\Gamma\, dr^2 - Y^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{11}$$

where $\Gamma = \Gamma(r, t)$, $Y = Y(r, t) > 0$. The situation is of interest because the metric coefficients do not have a factorized dependence of the coordinate variables. By using again the tabulated form for spherically symmetric space-time (Cahill and McVittie, 1970), we find that the independent nonzero components of the Riemann tensor take the values ($Y' = \partial Y/\partial r$)

$$R_{2323} = -Y^2 \sin^2\theta(1 + \dot{Y}^2 - Y'^2 e^{-\Gamma})$$

$$R_{1212} = \frac{R_{3131}}{\sin^2\theta} = Y\left(Y'' - \frac{\Gamma' Y'}{2} - \frac{\dot{Y}\dot{\Gamma}}{2}\, e^{\Gamma}\right)$$

$$R_{1220} = \frac{R_{1330}}{\sin^2\theta} = -Y\dot{Y}' + YY'\,\frac{\dot{\Gamma}}{2}$$

$$R_{1010} = e^{\Gamma}\left(\frac{\ddot{\Gamma}}{2} + \frac{\dot{\Gamma}^2}{4}\right) \tag{12}$$

$$R_{2020} = \frac{R_{3030}}{\sin^2\theta} = Y\ddot{Y}$$

To determine Ψ_{ABCD}, and in view of further considerations, we choose the null tetrad frame (Zecca, 1993)

$$l^i \equiv \frac{1}{\sqrt{2}}\,(1,\, e^{-\Gamma},\, 0,\, 0)$$

$$n^i \equiv \frac{1}{\sqrt{2}}\,(1,\, -e^{-\Gamma},\, 0,\, 0)$$

$$m^i \equiv \frac{1}{Y\sqrt{2}}\left(0,\, 0,\, 1,\, \frac{i}{\sin\theta}\right) \tag{13}$$

$$m^{*i} \equiv \frac{1}{Y\sqrt{2}}\left(0,\, 0,\, 1,\, -\frac{i}{\sin\theta}\right)$$

to which there correspond the σ-matrices

$$\sigma^t_{AA'} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^r_{AA'} = \frac{1}{2}\, e^{-\Gamma}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\sigma^\theta_{AA'} = \frac{1}{2Y}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^\varphi_{AA'} = \frac{i}{2Y\sin\theta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{14}$$

By proceeding as in the previous section, one gets after some calculations

$$X_{ABCD} = \sigma^\theta_{AB}\sigma^\theta_{CD}\left\{\frac{1}{4}\,(1 + \dot{Y}^2 - Y'^2 e^{-\Gamma}) + \frac{Y^2}{4}\left(\frac{\ddot{\Gamma}}{2} + \frac{\dot{\Gamma}^2}{4}\right)\right\}$$
$$+\ \sigma^t_{AB}\sigma^t_{CD}\left\{\frac{e^{-\Gamma}}{4Y}\left(Y'' - \frac{\Gamma' Y'}{2} - \frac{\dot{Y}\dot{\Gamma}}{2}\, e^{\Gamma}\right) - \frac{\ddot{Y}}{4Y}\right\}$$

$$+\ \sigma^r_{AB}\sigma^r_{CD}\left\{\frac{e^\Gamma \ddot{Y}}{4Y} - \frac{1}{4Y}\left(Y'' - \frac{\Gamma' Y'}{2} - \frac{\dot{Y}\dot{\Gamma}}{2}\, e^\Gamma\right)\right\} \tag{15}$$

(here $\dot{Y} = \partial Y/\partial t$, $Y' = \partial Y/\partial r$). By performing then the symmetrization $X_{A(BCD)}$, one arrives at

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$$

$$\Psi_2 = \frac{1}{24}\left\{\frac{1 + \dot{Y}^2 - Y'^2 e^{-\Gamma}}{Y^2} + \frac{e^{-\Gamma}}{Y}\left(Y'' - \frac{\Gamma' Y'}{2} - \frac{\dot{Y}\dot{\Gamma}}{2}\, e^\Gamma\right) + \frac{\ddot{\Gamma}}{2} + \frac{\dot{\Gamma}^2}{4} - \frac{\ddot{Y}}{Y}\right\} \tag{16}$$

The result can be checked in two limiting situations. By choosing $\exp(\Gamma) = R^2(t)/(1 - ar^2)$, $Y = rR(t)$ (the RW case) in Eq. (15) one obtains exactly the expression (8) for the spinor X_{ABCD}. On the other hand, let us consider the special case represented by the LTB cosmological model. For this model (which represents a spherically symmetric solution of the Einstein equation for a universe filled with dust matter), the Γ and Y functions are no longer independent, but are such that

$$\begin{aligned} e^\Gamma &= \frac{Y'^2}{1 + 2E} \\ \ddot{Y}Y^2 &= -m(r) \\ \frac{\dot{Y}^2}{2} - \frac{m(r)}{Y} &= E \\ m(r) &= 4\pi G \int_0^r \eta Y^2 Y'\, dr \end{aligned} \tag{17}$$

where $E = E(r)$ is an arbitrary function of r and $\eta = \eta(r, t)$ represents the density of the dust matter (see, e.g., Proceeding Kraziński, 1997, and references therein). When Eqs. (17) are put into Eq. (16), one gets

$$\Psi_2 = -\frac{1}{4}\left[-\frac{m(r)}{Y^3} + \frac{4}{3}\pi G\eta\right] \tag{18}$$

which is the expression obtained in Zecca (1993), apart from the $-1/4$ factor, which depends on the definition of the Weyl spinor used here (Penrose and Rindler, 1984) and there (Chandrasekhar, 1983).

It is worth noticing that the spinor component Ψ_2 in Eq. (16) depends in general on the two arbitrary functions Γ, Y. Also, by requiring the condition $\Psi_2 = 0$ (the space-time is then, at least locally, conformal to Minkowski space), the metric still depends on an arbitrary function. For example, consider the following cases:

(i) Suppose the metric coefficients have an RW-like form $Y = R(t)f(r)$, $e^{\Gamma} = R(t)g(r)$ with R a function of t, and f and g functions of r. By considering Eq. (16) with these assumptions, the equation $\Psi_2 = 0$ implies

$$\frac{1}{f^2} - \frac{1}{g}\left(\frac{f'^2}{f^2} - \frac{f''}{f}\right) - \frac{1}{2}\frac{g'}{g}\frac{\alpha'}{\alpha} = 0 \tag{19}$$

and $R(t)$ remains arbitrary. By choosing, e.g., $f = r$, or $f = \exp(\mp r)$, the expressions $g = 1 - ar^2$ and $g = [B + \exp(\pm 2r)]^{-1}$ (B is an integration constant) are then, respectively, solutions of Eq. (19).

(ii) Consider now the static case $\Gamma = \Gamma(r)$, $Y = Y(r)$ in Eq. (16). Then $\Psi_2 = 0$ implies, with $Z = -\exp(-\Gamma)$, the Bernoulli equation

$$-Z' + 2Z\left(\frac{Y''}{Y'} - \frac{Y'}{Y}\right) = -2\frac{Z^2}{YY'} \tag{20}$$

whose solution is

$$Z = -\left(\frac{2Y'^2}{Y^2}\int^r \frac{Y}{Y'^3}\,dr\right)^{-1}$$

If, for instance, $Y = r$, then $Z = -1$, $\Gamma = 0$.

4. LTB SPACE-TIME: THE $s = 3/2$ EQUATION

Equation (1) can be made explicit, in the Newman–Penrose formalism, by expanding in terms of the spin coefficients and of the directional derivatives $D = l^i\,\partial_i \equiv \partial_{00'}$, $\Delta = n^i\,\partial_i \equiv \partial_{11'}$, $\delta = m^i\,\partial_i \equiv \partial_{10'}$, $\delta^* = m^{*i}\partial_i \equiv \partial_{01'}$. In case of $s = 3/2$ one obtains, in a general space-time, the independent equations

$$\begin{aligned}
(D - 3\rho - \epsilon)\phi_1 - (\delta^* - 3\alpha + \pi)\phi_0 + 2\kappa\phi_2 &= 0\\
(D + \epsilon - 2\rho)\phi_2 - (\delta^* - \alpha + 2\pi)\phi_1 + \kappa\phi_3 + \lambda\phi_0 &= 0\\
(D - \rho + 3\epsilon)\phi_3 - (\delta^* + \alpha + 3\pi)\phi_2 + 2\lambda\phi_1 &= 0\\
(\Delta + \mu - 3\gamma)\phi_0 - (\delta - 3\tau - \beta)\phi_1 - 2\sigma\phi_2 &= 0\\
(\Delta + 2\mu - \gamma)\phi_1 - (\delta - 2\tau + \beta)\phi_2 - \sigma\phi_3 - \nu\phi_0 &= 0\\
(\Delta + 3\mu + \gamma)\phi_2 - (\delta - \tau + 3\beta)\phi_3 - 2\nu\phi_1 &= 0
\end{aligned} \tag{21}$$

The constraints (2) become for $s = 3/2$

$$\begin{aligned}
\phi_0\Psi_3 - 3\phi_1\Psi_2 + 3\phi_2\Psi_1 - \phi_3\Psi_0 &= 0\\
\phi_0\Psi_4 - 3\phi_1\Psi_3 + 3\phi_2\Psi_2 - \phi_3\Psi_1 &= 0
\end{aligned} \tag{22}$$

By considering the Weyl spinor in (16), the field solutions are therefore subject to the restrictions

$$\phi_1 = \phi_2 = 0 \tag{23}$$

In case of the LTB space-time, where the nonzero spin coefficients relative to the null tetrad (13) are (Zecca, 1993)

$$\begin{aligned} \rho &= -\frac{1}{Y\sqrt{2}}(\dot{Y} + Y'e^{-\Gamma/2}) \\ \mu &= \frac{1}{Y\sqrt{2}}(\dot{Y} - Y'e^{-\Gamma/2}) \\ \beta &= -\alpha = \frac{1}{2\sqrt{2}Y}\cot\theta \\ \epsilon &= -\gamma = \frac{\dot{\Gamma}}{4\sqrt{2}} \end{aligned} \tag{24}$$

the field equations (21) reduce to the system of differential equations

$$\begin{aligned} (\delta^* - 3\alpha)\phi_0 &= 0 \\ (\Delta + \mu - 3\gamma)\phi_0 &= 0 \\ (\delta - 3\alpha)\phi_3 &= 0 \\ (D + 3\epsilon - \rho)\phi_3 &= 0 \end{aligned} \tag{25}$$

By the structure of the directional derivatives and of the spin coefficients the angular part can be separated and integrated, for both ϕ_0 and ϕ_3, in the solution of Eqs. (25). The separated equations in the r, t variables can be rearranged so that one finally obtains

$$\begin{aligned} \phi_0 &= e^{im\varphi}(\sin\theta)^{-2}(\tan\theta/2)^{-m}\phi_0(r, t) \\ \phi_3 &= e^{-in\varphi}(\sin\theta)^{-2}(\tan\theta/2)^{-n}\phi_3(r, t) \end{aligned} \tag{26}$$

where $m, n = 0, \pm 1, \pm 2, \dots$, and $\phi_0(r, t)$ and $\phi_3(r, t)$ are solutions of the equations

$$\begin{aligned} \Delta[\log(Y\phi_0)] &= -\frac{3}{4\sqrt{2}}\dot{\Gamma} \\ \mathrm{D}[\log(Y\phi_3)] &= -\frac{3}{4\sqrt{2}}\dot{\Gamma} \end{aligned} \tag{27}$$

which could be solved, in principle, by giving the functions Γ, Y. It must be remarked that the solutions are not regular for $\theta = 0$.

5. LTB SPACE-TIME: THE $s = 2$ EQUATION

The independent massless spin-2 equations obtained by making Eq. (1) explicit as in the previous case are (Bell and Szekeres, 1972; Zecca, 1996)

$$\begin{aligned}
(D - 2\epsilon - 4\rho)\phi_1 - (\delta^* + \pi - 4\alpha)\phi_0 + 3\kappa\phi_2 &= 0 \\
(D - 3\rho)\phi_2 - (\delta^* + 2\pi - 2\alpha)\phi_1 + 2\kappa\phi_3 + \lambda\phi_0 &= 0 \\
(D - 2\epsilon - 2\rho)\phi_3 - (\delta^* + 3\pi)\phi_2 + \kappa\phi_4 + 2\lambda\phi_1 &= 0 \\
(D + 4\epsilon - \rho)\phi_4 - (\delta^* + 4\pi + 2\alpha)\phi_3 + 3\lambda\phi_2 &= 0 \\
(\Delta + \mu - 4\gamma)\phi_0 - (\delta - 4\tau - 2\beta)\phi_1 - 3\sigma\phi_2 &= 0 \\
(\Delta + 2\mu - 2\gamma)\phi_1 - (\delta - 3\tau)\phi_2 - 2\sigma\phi_3 - \nu\phi_0 &= 0 \\
(\Delta + 3\mu)\phi_2 - (\delta + 2\beta - 2\tau)\phi_3 - 2\nu\phi_1 - \sigma\phi_4 &= 0 \\
(\Delta + 4\mu + 2\gamma)\phi_3 - (\delta + 4\beta - \tau)\phi_4 + 3\nu\phi_2 &= 0
\end{aligned} \tag{28}$$

The constraints (2) for $s = 2$ become

$$\begin{aligned}
\phi_0\Psi_3 - 3\phi_1\Psi_2 + 3\phi_2\Psi_1 - \phi_3\Psi_0 &= 0 \\
\phi_1\Psi_4 - 3\phi_2\Psi_3 + 3\phi_3\Psi_2 - \phi_4\Psi_1 &= 0 \\
\phi_0\Psi_4 - 2\phi_1\Psi_3 + 3\phi_3\Psi_1 - \phi_4\Psi_0 &= 0
\end{aligned} \tag{29}$$

and together with (16) imply

$$\phi_1 = \phi_2 = 0 \tag{30}$$

By considering the spin coefficients (24) and the constraints (30) in Eqs. (29), we reduce the field equations to be solved to

$$\begin{aligned}
(\delta^* - 4\alpha)\phi_0 = (\Delta + \mu - 4\gamma)\phi_0 &= 0 \\
\delta\phi_2 = \delta^*\phi_2 &= 0 \\
(D - 3\rho)\phi_2 = (\Delta - 3\mu)\phi_2 &= 0 \\
(D - \rho - 4\epsilon)\phi_4 = (\delta + 4\beta)\phi_4 &= 0
\end{aligned} \tag{31}$$

From the second row in Eqs. (31) one has that ϕ_2 is independent of θ, φ, $\phi_2 = \phi_2\,(r, t)$, so that the two remaining equations for ϕ_2 can be simultaneously integrated to give

$$\phi_2 = \phi_2(r, t) = AY^{-3} \tag{32}$$

A is an integration constant. With regard to the equation relative to ϕ_0, ϕ_4, by considering the expressions of the spin coefficients and of the directional derivatives, one has that the angular part factors out and can be easily integrated as in the $s = 3/2$ case. The radial equations can be recast into a simpler form so that the solutions are

$$\begin{aligned} \phi_0 &= e^{im\varphi}(\sin\theta)^{-3/2}(\tan\theta/2)^{-m}\phi_0(r, t) \\ \phi_4 &= e^{-in\varphi}(\sin\theta)^{-3/2}(\tan\theta/2)^{-n}\phi_4(r, t) \end{aligned} \tag{33}$$

$m, n = 0, \pm 1, \pm 2, \ldots$, and $\phi_0(r, t)$ and $\phi_4(r, t)$ are determined by the equations

$$\begin{aligned} \Delta[\log(Y\phi_0)] &= -\frac{\dot{\Gamma}}{\sqrt{2}} \\ D[\log(Y\phi_4)] &= -\frac{\dot{\Gamma}}{\sqrt{2}} \end{aligned} \tag{34}$$

Also here the solutions are not regular for $\theta = 0$.

6. LTB SPACE-TIME: GENERALIZATIONS

The previous conclusions can be extended to the general case of arbitrary spin. The system of coupled differential equations obtained by developing Eqs. (1) in terms of the directional derivatives and of the spin coefficients seems to be very difficult to solve in a general curved space-time. However, owing to the particular structure of the spin coefficients (24) one immediately realizes that the considerations developed for the RW space-time in Zecca (1996) apply also to the LTB case. Therefore Eqs. (1), when expanded in the LTB case, become a system of coupled equations each of which involves only two components ϕ_h, ϕ_{h+1} of the field. [This is evident in Eqs. (21), (28) by using the fact that in LTB space-time $\kappa = \lambda = \pi = \tau = \sigma = \nu = 0$, and the property follows by induction in the general case (Zecca, 1996).] On the other hand, when the constraints (2) are taken into consideration together with the form (16) of the Weyl spinor, one must also require $\phi_h = 0$ for a suitable set of values of h depending on n. More precisely, it is not difficult to show that Eqs. (2) and (16) imply the following possibilities:

1. s is integer; then $\phi_h = 0$ for $h \neq 0, (n + 1)/2, n + 1$.
2. s is half-integer; then $\phi_h = 0$ for $h \neq 0, n + 1$.

Therefore the solution of the massless free field equations for arbitrary spin in the LTB space-time reduce to the determination of the two or three nonzero independent components of the field according to whether the value of the spin is half-integer or integer. This can be performed in the same fashion as was previously done for $s = 3/2$ and $s = 2$, respectively.

ACKNOWLEDGMENT

It is a pleasure to thank Dr. Sanjay Jhingan for calling my attention to useful references.

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